Scattering Theory in the Energy Space for a Class of Hartree Equations*

Dedicated to Professor Walter A. Strauss on his 60th birthday

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Abstract

We study the theory of scattering in the energy space for the Hartree equation in space dimension $n \geq 3$. Using the method of Morawetz and Strauss, we prove in particular asymptotic completeness for radial nonnegative nonincreasing potentials satisfying suitable regularity properties at the origin and suitable decay properties at infinity. The results cover in particular the case of the potential $|x|^{-\gamma}$ for $2 < \gamma < \mathrm{Min}(4, n)$.

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1 Introduction.

This paper is devoted to the theory of scattering for the Hartree equation

$$i\partial_t u = -(1/2)\Delta u + u\left(V \star |u|^2\right) \quad . \tag{1.1}$$

Here u is a complex valued function defined in space time $I\!R^{n+1}$, Δ is the Laplacian in $I\!R^n$, V is a real valued even function defined in $I\!R^n$, hereafter called the potential, and $\star$ denotes the convolution in $I\!R^n$. In particular we develop a complete theory of scattering for the equation (1.1) in the energy space, which turns out to be the Sobolev space H^1, under suitable assumptions on V.

One of the basic problems addressed by scattering theory is that of classifying the asymptotic behaviours in time of the global solutions of a given evolution equation by comparing them with those of the solutions of a suitably chosen and simpler evolution equation. In the special case where the given equation is a (nonlinear) perturbation of a linear dispersive equation, the first obvious candidate as a comparison equation is the underlying linear equation, hereafter called the free equation, and we restrict the discussion to that case. In the case of the Hartree equation (1.1), that equation is the free Schrödinger equation

$$i\partial_t u = -(1/2)\Delta u \quad . \tag{1.2}$$

Let $U(t)$ be the evolution group that solves the free equation. The comparison between the two equations then gives rise to the following two basic questions.

(1) Existence of the wave operators. For any solution $v_+(t) = U(t)\ u_+$ of the free equation with initial data $v_+(0) = u_+$, called the asymptotic state, in a suitable Banach space Y, one looks for a solution u of the original equation which behaves asymptotically as v_+ when $t \to \infty$, typically in the sense that

$$\| \, u(t) - v_+(t); Y \, \| \to 0 \qquad \text{when } t \to +\infty \tag{1.3}$$

or rather

$$\| \, U(-t)u(t) - u_+; Y \, \| \to 0 \qquad \text{when } t \to +\infty \tag{1.4}$$

which may be more appropriate if $U(\cdot)$ is not a bounded group in Y. If such a u can be constructed for any $u_+ \in Y$, one defines the wave operator Ω_+ for positive time as the map

$u_+ \to u(0)$. The same problem arises at $t \to -\infty$, thereby leading to the definition of the wave operator Ω_- for negative time.

(2) Asymptotic completeness. Conversely, given a solution u of the original equation, one looks for asymptotic states u_+ and u_- such that $v_\pm(t) = U(t)\ u_\pm$ behaves asymptotically as $u(t)$ when $t \to \pm\infty$, typically in the sense that (1.3) or (1.4) and their analogues for negative time hold. If that can be realized for any u with initial data $u(0)$ in Y for some $u_\pm \in Y$, one says that asymptotic completeness holds in Y.

Asymptotic completeness is a much harder problem than the existence of the wave operators, except in the case of small data where it follows as an immediate by-product of the method generally used to solve the latter problem. Asymptotic completeness for large data in the sense described above requires strong assumptions on the nonlinear perturbation, in particular some repulsivity condition, and proceeds through the derivation of a priori estimates for general solutions of the original equation. It has been derived so far only for a small number of nonlinear evolution equations.

Scattering theory for nonlinear evolution equations started in the sixties under the impulse of Segal [24] [25], on the example of the nonlinear wave (NLW) equation

$$\Box u + f(u) = 0 \tag{1.5}$$

and of the nonlinear Klein-Gordon (NLKG) equation

$$\left(\Box + m^2\right) u + f(u) = 0 \tag{1.6}$$

with $f(u)$ a nonlinear interaction term, a typical form of which is

$$f(u) = |u|^{p-1} u \tag{1.7}$$

for some $p > 1$. Following early works where asymptotic completeness was ensured by including an explicit integrable decay in time or a suitable decay in space in the nonlinearity (see references quoted in [26]), major contributions were made by Strauss on the NLW equation [26] and by Morawetz and Strauss on the NLKG equation [22]. In [26] a complete theory of scattering is developed for the NLW equation (1.5) in space dimension $n = 3$, in a space of suitably regular and decaying functions, for a large class of nonlinearities including the form (1.7) under the

natural assumption $3 \le p < 5$ (more generally $4/(n-1) \le p-1 < 4/(n-2)$ in space dimension n). Asymptotic completeness is proved there by exploiting the approximate conservation law associated with the approximate conformal invariance of the equation. In [22], a complete theory of scattering is developed for the NLKG equation (1.6) in space dimension $n = 3$ with nonlinear interaction (1.7) and $p = 3$. Asymptotic completeness is proved there by the use of the Morawetz inequality [21]. That inequality however is not a very strong statement, since it asserts only the convergence of a space time integral which would naively be expected to be barely divergent, and it required a tour de force in analysis to extract therefrom the necessary a priori estimates. The method of [22] was then extended by Lin and Strauss [20] to construct a complete theory of scattering for the nonlinear Schrödinger (NLS) equation

$$i\partial_t u = -(1/2)\Delta u + f(u) \tag{1.8}$$

in space dimension $n = 3$ with nonlinear interaction (1.7) and $8/3 < p < 5$. Parallel and subsequent developments included the construction of a complete theory of scattering in the space $\Sigma = H^1 \cap \mathcal{F}H^1$, where H^1 is the usual Sobolev space and $\mathcal{F}$ the Fourier transform, in arbitrary space dimension, both for the NLS equation (1.8) [9] [17] [31] and for the Hartree equation (1.1) [10] [18] [23]. Asymptotic completeness is proved there by the use of the approximate conservation law associated with the approximate pseudo-conformal invariance of the NLS and Hartree equations, which is the analogue for those equations of the conformal invariance of the NLW equation exploited in [26]. The class of interactions thereby covered includes the form (1.7) for the NLS equation (1.8) with

$$p_0(n) < p < 1 + 4/(n-2) \quad , \tag{1.9}$$

where $p_0(n)$ is the positive root of the equation $np(p-1) = 2(p+1)$, and includes the potential

$$V(x) = C\ |x|^{-\gamma} \tag{1.10}$$

with $C > 0$ and $4/3 < \gamma < \mathrm{Min}(4, n)$ for the Hartree equation (1.1).

Meanwhile the paper [22] inspired further developments. It turned out that the natural function space of initial data and asymptotic states for the implementation of the method of [22] is the energy space, and a complete theory of scattering in that space was constructed for

the NLKG equation (1.6) in arbitrary dimension $n \geq 3$, under assumptions on f which in the special case (1.7) reduce to the natural condition

$$1 + 4/n < p < 1 + 4/(n-2) \tag{1.11}$$

[5] [6], see also [12]. A complete theory of scattering in the energy space, in that case the Sobolev space H^1, was then constructed for the NLS equation in dimension $n \geq 3$ again by the use of a variant of the method of [22], under assumptions on f which in the special case (1.7) again reduce to the natural condition (1.11) [11] [12]. The construction was then somewhat simplified in [7] and in [8]. Finally, the method of [22] was extended to construct a complete theory of scattering in the energy space for the NLW equation (1.5) under assumptions on f which barely miss the special case (1.7) with $p = 1 + 4/(n-2)$, the H^1 critical value [13]. However the proof of asymptotic completeness by that method for that equation is now superseded by more direct estimates which cover that critical case [1] [2] [3]. Expositions of the theory at various stages of its development can be found in [27] [28] [30].

In this paper, we develop a complete theory of scattering for the Hartree equation (1.1) in the energy space, which is again the Sobolev space H^1. The essential part of that theory is the proof of asymptotic completeness, which is obtained by an adaptation of the method of [22]. As a preliminary, we present briefly the theory of the Cauchy problem at finite times and the construction of the wave operators in the energy space H^1. That part of the theory is a simple variant of the corresponding theory for the NLS equation, which has reached a well developed stage [7] [8] [19]. Consequently, although we give complete statements of the results, we provide only brief sketches of the proofs in that part. The exposition follows closely that in [8] for the NLS equation.

In all this paper, we restrict our attention to space dimension $n \geq 3$, because the method of [22] applies only to that case. Most of the results on the Cauchy problem at finite times and on the existence of the wave operators would extend to lower dimensions, where they would however require modified statements.

The assumptions made on V for the Cauchy problem at finite times and for the existence of the wave operators are of a general character. In the typical situation where $V \in L^p$ for some p, $1 \leq p \leq \infty$, they reduce to the condition $p > n/4$ for the Cauchy problem at finite times and

to $n/4 < p \leq n/2$ for the existence of the wave operators, thereby covering the example (1.10) for $\gamma < \mathrm{Min}(4, n)$ and $2 < \gamma < \mathrm{Min}(4, n)$ respectively. On the other hand stronger assumptions are required for the proof of asymptotic completeness. In particular the potential V should in addition be radial and suitably repulsive (see Assumption (H3) in Section 4 below). Those requirements are satisfied in particular by the special case (1.10), and a complete theory can be developed for that case for $2 < \gamma < \mathrm{Min}(4, n)$.

This paper is organized as follows. In Section 2 we treat the Cauchy problem at finite times for the equation (1.1). We prove local wellposedness in H^1 (Proposition 2.1), we derive the conservation laws of the L^2 norm and of the energy (Proposition 2.2), and we prove global wellposedness in H^1 (Proposition 2.3). In Section 3, we prove the existence of the wave operators. We solve the local Cauchy problem in a neighbourhood of infinity in time (Proposition 3.1), we prove the existence and some properties of asymptotic states for the solutions thereby obtained (Proposition 3.2), and we conclude with the existence of the wave operators (Proposition 3.3). Finally in Section 4, we prove the main result of this paper, namely asymptotic completeness in H^1. We derive the Morawetz inequality for the Hartree equation (1.1) (Proposition 4.1), we extract therefrom an estimate of the solutions in suitable norms (Proposition 4.2), and we exploit that estimate to prove asymptotic completeness (Proposition 4.3). A more detailed description of the contents of Section 4 is provided at the beginning of that section.

We conclude this introduction by giving some notation which will be used freely throughout this paper. For any integer $n \geq 3$, we let $2^\star = 2n/(n-2)$. For any r, $1 \leq r \leq \infty$, we denote by $\| \cdot \|_r$ the norm in $L^r \equiv L^r(I\!R^n)$ and by $\bar{r}$ the conjugate exponent defined by $1/r + 1/\bar{r} = 1$, and we define $\delta(r) = n/2 - n/r$. We denote by $< \cdot, \cdot >$ the scalar product in L^2 and by H^1_r the Sobolev space

$$H^1_r = \left\{ u : \| u; H^1_r \| = \| u \|_r \ + \ \| \nabla u \|_r < \infty \right\} \quad .$$

For any interval I of $I\!R$, for any Banach space X, we denote by $\mathcal{C}(I, X)$ the space of continuous functions from I to X and for $1 \leq q \leq \infty$, by $L^q(I, X)$ (resp. $L^q_{loc}(I, X)$) the space of measurable functions u from I to X such that $\| u(\cdot); X \| \in L^q(I)$ (resp. $\in L^q_{loc}(I)$). For any interval I of $I\!R$, we denote by $\bar{I}$ the closure of I in $\bar{I\!R} = I\!R \cup \{\pm\infty\}$ equipped with the natural topology. Finally for any real numbers a and b, we let $a \vee b = \mathrm{Max}(a, b)$, $a \wedge b = \mathrm{Min}(a, b)$, $a_+ = a \vee 0$

and $a_- = (-a)_+$.

2 The Cauchy problem at finite times.

In this section, we briefly recall the relevant results on the Cauchy problem with finite initial time for the equation (1.1). We refer to [7] [10] for more details. We rewrite the equation (1.1) as

$$i\partial_t u = -(1/2)\Delta u + f(u) \tag{2.1}$$

where

$$f(u) = u\left(V \star |u|^2\right) \quad . \tag{2.2}$$

The Cauchy problem for the equation (2.1) with initial data $u(0) = u_0$ at $t = 0$ is rewritten in the form of the integral equation

$$u(t) = U(t)\ u_0 - i\int_0^t dt'\ U(t-t')\ f(u(t')) \tag{2.3}$$

where $U(t)$ is the unitary group

$$U(t) = \exp\left(it\Delta/2\right) \quad .$$

It is well known that $U(t)$ satisfies the pointwise estimate

$$\parallel U(t)f \parallel_r \le (2\pi|t|)^{-\delta(r)} \ \parallel f \parallel_{\bar{r}} \tag{2.4}$$

where $2 \le r \le \infty$, $1/r + 1/\bar{r} = 1$ and $\delta(r) \equiv n/2 - n/r$.

Let now I be an interval, possibly unbounded. We define the operators

$$(U \star f)(t) = \int_I dt'\ U(t-t')\ f(t') \quad , \tag{2.5}$$

$$(U \star_R f)(t) = \int_{I\cap\{t'\le t\}} dt'\ U(t-t')\ f(t') \quad , \tag{2.6}$$

where f is defined in $I\!R^n \times I$ and suitably regular, the dependence on I is omitted and the subscript R stands for retarded. We introduce the following definition

Definition 2.1. A pair of exponents (q, r) is said to be admissible if

$$0 \le 2/q = \delta(r) < 1 \quad .$$

It is well known that $U(t)$ satisfies the following Strichartz estimates [7] [8] [19].

Lemma 2.1. *The following estimates hold :*

(1) For any admissible pair (q,r)

$$\| U(t)\ u; L^q(I\!R, L^r) \| \le c_r \ \ \| u \|_2 \quad . \tag{2.7}$$

(2) For any admissible pairs (q_i, r_i)*,* $i = 1, 2,$ *and for any interval* $I \subset I\!R$

$$\| U \star f; L^{q_1}(I, L^{r_1}) \| \le c_{r_1}\ c_{r_2} \ \ \| f; L^{\bar{q}_2}(I, L^{\bar{r}_2}) \| \quad , \tag{2.8}$$

$$\| U \star_R f; L^{q_1}(I, L^{r_1}) \| \le c_{r_1}\ c_{r_2} \ \ \| f; L^{\bar{q}_2}(I, L^{\bar{r}_2}) \| \quad . \tag{2.9}$$

Lemma 2.1 suggests that we study the Cauchy problem for the equation (2.3) in spaces of the following type. Let I be an interval. We define

$$X(I) = \Big\{ u : u \in \mathcal{C}(I, L^2) \text{ and } u \in L^q(I, L^r) \text{ for } 0 \le 2/q = \delta(r) < 1 \Big\} \quad , \tag{2.10}$$

$$X^1(I) = \{ u : u \text{ and } \nabla u \in X(I) \} \quad . \tag{2.11}$$

For noncompact I, we define the spaces $X_{loc}(I)$ and $X^1_{loc}(I)$ in a similar way by replacing L^q by L^q_{loc} in (2.10).

The spaces $X(I)$ and $X^1(I)$ are not Banach spaces in a natural way because the interval for r in (2.10) is semi-open. We define Banach spaces by restricting the interval for r by $0 \le \delta(r) \le 2/q_0 = \delta(r_0) \equiv \delta_0 < 1$, namely

$$\begin{aligned} X_{r_0}(I) &= \Big\{ u : u \in \mathcal{C}(I, L^2) \text{ and } u \in L^q(I, L^r) \text{ for } 0 \le 2/q = \delta(r) \le \delta_0 \Big\} \\ &= (\mathcal{C} \cap L^\infty)(I, L^2) \cap L^{q_0}(I, L^{r_0}) \quad , \qquad (2.12) \\ X^1_{r_0}(I) &= \{ u : u \text{ and } \nabla u \in X_{r_0}(I) \} \quad . \qquad (2.13) \end{aligned}$$

The spaces $X_{r_0,loc}(I)$ and $X^1_{r_0,loc}(I)$ are defined in a natural way.

In all this paper, we assume that the potential V satisfies the following assumption.

(H1) V is a real even function and $V \in L^{p_1} + L^{p_2}$ for some p_1, p_2 satisfying

$$1 \vee (n/4) \le p_2 \le p_1 \le \infty \quad . \tag{2.14}$$

We can now state the main result on the local Cauchy problem for the equation (1.1) with H^1 initial data.

Proposition 2.1. *Let V satisfy (H1) and define r_0 by*

$$\delta_0 \equiv \delta(r_0) = (n/4p_2 - 1/2)_+ \quad (\leq 1/2) \quad . \tag{2.15}$$

Let $u_0 \in H^1$. Then

(1) There exists a maximal interval $(-T_-, T_+)$ with $T_\pm > 0$ such that the equation (2.3) has a unique solution $u \in X^1_{r_0,loc}((-T_-, T_+))$. The solution u actually belongs to $X^1_{loc}((-T_-, T_+))$.

(2) For any interval I containing 0, the equation (2.3) has at most one solution in $X^1_{r_0}(I)$.

(3) For $-T_- < T_1 \leq T_2 < T_+$, the map $u_0 \to u$ is continuous from H^1 to $X^1([T_1, T_2])$.

(4) Let in addition $p_2 > n/4$. Then if $T_+ < \infty$ (resp. $T_- < \infty$), $\| u(t); H^1 \| \to \infty$ when t increases to T_+ (resp. decreases to $-T_-$).

Sketch of proof. The proof proceeds by standard arguments. The main technical point consists in proving that the operator defined by the RHS of (2.3) is a contraction in $X^1_{r_0}(I)$ on suitable bounded sets of $X^1_{r_0}(I)$ for $I = [-T, T]$ and T sufficiently small. The basic estimates follow from Lemma 2.1 and from

$$\| f(u); L^{\bar{q}}(I, H^1_{\bar{r}}) \| \leq C \| V \|_p \| u; L^q(I, H^1_r) \| \| u; L^k(I, L^s) \|^2 \ T^\theta \tag{2.16}$$

where we have assumed for simplicity that $V \in L^p$, where (q, r) is an admissible pair, and where the exponents satisfy

$$n/p = 2\delta(r) + 2\delta(s) \tag{2.17}$$

$$2/q + 2/k = 1 - \theta \quad . \tag{2.18}$$

The estimate (2.16) is obtained by applying the Hölder and Young inequalities in space followed by the Hölder inequality in time, which requires $0 \leq \theta \leq 1$. A similar estimate holds for the difference of two solutions. For general V satisfying (H1), the contribution of the components in L^{p_1} and L^{p_2} are treated separately, with the exponents (q, r, k, s) possibly depending on p.

If $n/p \leq 2$, one can choose $r = 2$, $\delta(s) = n/2p \leq 1$, $k = q = \infty$, so that $\theta = 1$ and one can take $r_0 = 2$.

If $n/p \geq 2$, one can take $\delta(s) = \delta(r)+1$ and $k = q$, so that $4/q = 1-\theta$ and $n/p = 4\delta(r)+2 = 4 - 2\theta$, which yields $\theta \geq 0$ for $n/p \leq 4$, and allows for $\delta_0 = n/4p - 1/2$.

The H^1-critical case $p = n/4$ yields $\theta = 0$ and requires a slightly more refined treatment than the subcritical case $p > n/4$.

$\square$

It is well known that the Hartree equation (1.1) formally satisfies the conservation of the L^2 norm and of the energy

$$E(u) = \frac{1}{2} \parallel \nabla u \parallel_2^2 \ + \frac{1}{2} \int dx\ dy\ |u(x)|^2\ V(x-y)\ |u(y)|^2 \quad . \tag{2.19}$$

Actually it turns out that the X^1 regularity of the solutions constructed in Proposition 2.1 is sufficient to ensure those conservation laws.

Proposition 2.2. *Let V satisfy (H1) and define r_0 by (2.15). Let I be an interval and let $u \in X^1_{r_0}(I)$ be a solution of the equation (1.1). Then u satisfies $\parallel u(t_1) \parallel_2 = \parallel u(t_2) \parallel_2$ and $E(u(t_1)) = E(u(t_2))$ for all t_1, $t_2 \in I$.*

Sketch of proof. We consider only the more difficult case of the energy and we follow the proof of the corresponding result for the nonlinear Schrödinger (NLS) equation given in [8]. Let $\varphi \in \mathcal{C}_0^\infty$ be a smooth approximation of the Dirac distribution δ in $I\!R^n$. By an elementary computation which is allowed by the available regularity, one obtains

$$\begin{aligned} &E\left((\varphi \star u)(t_2)\right) - E\left((\varphi \star u)(t_1)\right) = -\mathrm{Im} \int_{t_1}^{t_2} dt \Big\{ < \varphi \star \nabla u, \nabla f(\varphi \star u) - \varphi \star \nabla f(u) > \\ &+2 < \varphi \star f(u), f(\varphi \star u) - \varphi \star f(u) > \Big\}(t) \quad . \end{aligned} \tag{2.20}$$

One then lets φ tend to δ, using the fact that convolution with φ tends strongly to the unit operator in L^r for $1 \leq r < \infty$. The LHS of (2.20) tends to $E(u(t_2)) - E(u(t_1))$ and the RHS is shown to tend to zero by the Lebesgue dominated convergence theorem applied to the time integration. For that purpose one needs an estimate of the integrand which is uniform in φ and

integrable in time. That estimate essentially boils down to

$$| < \nabla u, \nabla f > | \ \leq C \parallel V \parallel_p \parallel \nabla u \parallel_r^2 \parallel u \parallel_s^2 \tag{2.21}$$

with r and s satisfying (2.17) and to

$$\parallel f(u) \parallel_2^2 \leq C \parallel V \parallel_p^2 \parallel u \parallel_s^6 \tag{2.22}$$

with $\delta(s) = n/3p$, for a potential $V \in L^p$.

For (2.21), we choose the same values of r, s as in the proof of Proposition 2.1, so that the RHS of (2.21) belongs to L^∞ in time for $n/p \leq 2$ and to $L^{q/4}$ with $q \geq 4$ for $n/p \geq 2$. On the other hand, the RHS of (2.22) belongs to L^∞ in time for $n/p \leq 3$ and to $L^{q/6}$ with $2/q = \delta(s) - 1 = n/3p - 1 \leq 1/3$ for $3 \leq n/p \leq 4$.

$\square$

We now turn to the global Cauchy problem for the equation (1.1). For that purpose we need to ensure that the conservation of the L^2 norm and of the energy provides an a priori estimate of the H^1 norm of the solution. This is the case if the potential V satisfies the following assumption

(H2) $V_- \equiv V \wedge 0 \in L^{n/2} + L^\infty$.

In fact, it follows from (H2) by the Hölder, Young and Sobolev inequalities that for any $\varepsilon > 0$, there exists $C(\varepsilon)$ such that

$$\int dx \ dy \ |u(x)|^2 \ V_-(x-y)|u(y)|^2 \leq \varepsilon \parallel u \parallel_2^2 \parallel \nabla u \parallel_2^2 + C(\varepsilon) \parallel u \parallel_2^4 \tag{2.23}$$

and therefore

$$\parallel \nabla u \parallel_2^2 \leq 4E(u) + 2C \left((2 \parallel u \parallel_2^2)^{-1} \right) \ \parallel u \parallel_2^4 \quad . \tag{2.24}$$

We can now state the main result on the global Cauchy problem for the equation (1.1).

Proposition 2.3. *Let V satisfy (H1) with $p_2 > n/4$ and (H2). Let $u_0 \in H^1$ and let u be the solution of the equation (2.3) constructed in Proposition 2.1. Then $T_+ = T_- = \infty$ and $u \in X^1_{loc}(I\!R) \cap L^\infty(I\!R, H^1)$.*

The proof is standard. Note however that the result is stated only for the H^1 subcritical case $p_2 > n/4$.

3 Scattering Theory I. Existence of the wave operators.

In this section we begin the study of the theory of scattering for the Hartree equation (1.1) by addressing the first question raised in the introduction, namely that of the existence of the wave operators. We restrict our attention to positive time. We consider an asymptotic state $u_+ \in H^1$ and we look for a solution u of the equation (1.1) which is asymptotic to the solution $v(t) = U(t)u_+$ of the free equation. For that purpose, we introduce the solution u_{t_0} of the equation (1.1) satisfying the initial condition $u_{t_0}(t_0) = v(t_0) \equiv U(t_0)u_+$. We then let t_0 tend to ∞. In favourable circumstances, we expect u_{t_0} to converge to a solution u of the equation (1.1) which is asymptotic to $v(t)$. The previous procedure is easily formulated in terms of integral equations. The Cauchy problem with initial data $u(t_0)$ at time t_0 is equivalent to the equation

$$u(t) = U(t-t_0)\ u(t_0) - i \int_{t_0}^{t} dt'\ U(t-t')\ f(u(t')) \quad . \tag{3.1}$$

The solution u_{t_0} with initial data $U(t_0)u_+$ at time t_0 should therefore be a solution of the equation

$$u(t) = U(t)u_+ - i \int_{t_0}^{t} dt'\ U(t-t')\ f(u(t')) \quad . \tag{3.2}$$

The limiting solution u is then expected to satisfy the equation

$$u(t) = U(t)\ u_+ + i \int_{t}^{\infty} dt'\ U(t-t')\ f(u(t')) \quad . \tag{3.3}$$

The problem of existence of the wave operators is therefore the Cauchy problem with infinite initial time. We solve that problem in two steps. We first solve it locally in a neighborhood of infinity by a contraction method. We then extend the solutions thereby obtained to all times by using the available results on the Cauchy problem at finite times. In order to solve the local Cauchy problem at infinity, we need to use function spaces including some time decay in their definition, so that at the very least the integral in (3.3) converges at infinity. Furthermore the free solution $U(t)u_+$ should belong to those spaces. In view of Lemma 2.1, natural candidates are the spaces $X^1_{r_0}(I)$ for some $I = [T, \infty)$, where the time decay is expressed by the L^q integrability at infinity, and we shall therefore study that problem in those spaces. We shall

also need the fact that the time decay of u implies sufficient time decay of $f(u)$. This will show up through additional assumptions on V in the form of an upper bound on p_1, namely $p_1 \leq n/2$.

For future reference we state additional time integrability properties of functions in $X^1_{r_0}(\cdot)$ which are not immediately apparent on the definition.

Lemma 3.1. *Let I be an interval, possibly unbounded. Then*

$$\| u; L^{q_0}(I, L^r) \| \leq C \| u; X^1_{r_0}(I) \| \tag{3.4}$$

for $\delta_0 \leq \delta(r) \leq \delta_0 + 1$, where C is independent of I.

Proof. The result follows from the Sobolev inequality

$$\| u \|_r \leq C \| u \|^{1-\sigma}_{r_0} \| \nabla u \|^{\sigma}_{r_0}$$

with $\sigma = \delta(r) - \delta(r_0)$ and from the definition of $X^1_{r_0}$.

□

We shall use freely the notation $\tilde{u}(t) = U(-t)u(t)$ for u a suitably regular function of space time. We also recall the notation $\overline{I\!R}$ for $I\!R \cup \{\pm\infty\}$ and $\bar{I}$ for the closure of an interval I in $\overline{I\!R}$ equipped with the obvious topology.

We can now state the main result on the local Cauchy problem in a neighborhood of infinity.

Proposition 3.1. *Let $r_0 = 2n/(n-1)$, so that $\delta_0 = 1/2$. Let V satisfy (H1) with $p_1 \leq n/2$. Let $u_+ \in H^1$. Then*

(1) There exists $T < \infty$ such that for any $t_0 \in \bar{I}$ where $I \in [T, \infty)$, the equation (3.2) has a unique solution u in $X^1_{r_0}(I)$. The solution u actually belongs to $X^1(I)$.

(2) For any $T' > T$, the solution u is strongly continuous from $u_+ \in H^1$ and $t_0 \in \bar{I}'$ to $X^1(I')$, where $I' = [T', \infty)$.

Sketch of proof. The proof proceeds by a contraction argument in $X^1_{r_0}(I)$. The main technical point consists in proving that the operator defined by the RHS of (3.2) is a contraction in $X^1_{r_0}(I)$

on suitable bounded sets of $X^1_{r_0}(I)$ for T sufficiently large. The basic estimate is again (2.16) supplemented by (2.17) (2.18), now however with $\theta = 0$. The fact that we use spaces where the time decay appears in the form of an L^q integrability condition in time forces the condition $\theta = 0$, so that we are in a critical situation, as was the case for the local Cauchy problem at finite times in the H^1 critical case $p_2 = n/4$. We choose the exponents in (2.16) as follows. We take $q = k = q_0 = 4$, $\delta(r) = \delta_0 = 1/2$ and $1 + 2\delta(s) = n/p$ so that the last norm in (2.16) is controlled by the $X^1_{r_0}$ norm for $1/2 \le \delta(s) \le 3/2$, namely $2 \le n/p \le 4$. The smallness condition which ensures the contraction takes the form

$$\| \, U(t) \, u_+ ; L^{q_0}(I, H^1_{r_0}) \, \| \le R_0 \tag{3.5}$$

for some absolute constant R_0. In particular the time T of local resolution cannot be expressed in terms of the H^1 norm of u_+ alone, as is typical of a critical situation.

The continuity in t_0 up to and including infinity follows from an additional application of the same estimates.

□

An immediate consequence of the estimates in the proof of Proposition 3.1 is the existence of asymptotic states for solutions of the equation (1.1) in $X^1_{r_0}([T,\infty))$ for some T. Furthermore the conservation laws of the L^2 norm and of the energy are easily extended to infinite time for such solutions.

Proposition 3.2. *Let $r_0 = 2n/(n-1)$. Let V satisfy (H1) with $p_1 \le n/2$. Let $T \in I\!R$, $I = [T, \infty)$ and let $u \in X^1_{r_0}(I)$ be a solution of the equation (1.1). Then*

(1) $\tilde{u} \in \mathcal{C}(\bar{I}, H^1)$. In particular the following limit exists

$$\tilde{u}(\infty) = \lim_{t\to\infty} \tilde{u}(t) \tag{3.6}$$

as a strong limit in H^1.

(2) u satisfies the equation (3.3) with $u_+ = \tilde{u}(\infty)$.

(3) u satisfies the conservation laws

$$\| \, \tilde{u}(\infty) \, \|_2 = \| \, u \, \|_2 \quad , \quad (1/2) \, \| \, \nabla \tilde{u}(\infty) \, \|_2^2 = E(u) \quad . \tag{3.7}$$

Sketch of proof. Part (1). We estimate for $T \le t_1 \le t_2$

$$\begin{aligned} \| \widetilde{u}(t_2) - \widetilde{u}(t_1); H^1 \| &= \| \int_{t_1}^{t_2} dt\ U(t_2 - t)\ f(u(t)); H^1 \| \\ &\le \| U \star f; X^1_{r_0}([t_1, t_2]) \| \le C \| f(u); L^{\bar{q}_0}([t_1, t_2]; H^1_{\bar{r}_0}) \| \end{aligned} \tag{3.8}$$

and we estimate the last norm as in the proof of Proposition 3.1, namely by (2.16) with $\theta = 0$ and with the same choice of exponents.

Part (2) follows from Part (1) and from Proposition 3.1, especially part (2).

Part (3). From the conservation laws at finite time and from Part (1), it follows that the following limits exist

$$\begin{aligned} \| \widetilde{u}(\infty) \|_2 &= \lim_{t\to\infty} \| u(t) \|_2 = \| u \|_2 \quad , \\ \lim_{t\to\infty} P(u(t)) &= E(u) - (1/2) \lim_{t\to\infty} \| \nabla\widetilde{u}(t) \|_2^2 \\ &= E(u) - (1/2) \| \nabla\widetilde{u}(\infty) \|_2^2 \end{aligned} \tag{3.9}$$

where

$$P(u) = \frac{1}{2} \int dx\ dy |u(x)|^2\ V(x - y)\ |u(y)|^2 \quad . \tag{3.10}$$

On the other hand

$$|P(u)| \le C \| V \|_p \| u \|_r^4 \in L^{q_0/4} = L^1 \tag{3.11}$$

by the Hölder and Young inequalities and by Lemma 3.1 with $\delta_0 = 1/2 \le \delta(r) = n/4p \le 1$.

It then follows from (3.11) that the limit in (3.9) is zero.

$\square$

The existence and the properties of the wave operators now follow from the previous local results at infinity and from the global results of Section 2.

Proposition 3.3. *Let $r_0 = 2n/(n-1)$. Let V satisfy (H1) and (H2) with $p_1 \le n/2$ and $p_2 > n/4$. Then*

(1) For any $u_+ \in H^1$, the equation (3.3) has a unique solution u in $X^1_{r_0,loc}(I\!R)$ with restriction in $X^1_{r_0}(I\!R^+)$. In addition, $u \in X^1_{loc}(I\!R)$ with restriction in $X^1(I\!R^+)$, and $\widetilde{u} \in \mathcal{C}(I\!R \cup \{+\infty\}, H^1)$. Furthermore u satisfies the conservation laws

$$\| u(t) \|_2 = \| u_+ \|_2 \qquad , \quad E(u(t)) = (1/2) \| \nabla u_+ \|_2^2$$

for all $t \in I\!R$.

(2) The wave operator $\Omega_+ : u_+ \to u(0)$ is well defined in H^1, and is continuous and bounded in the H^1 norm.

Sketch of proof. Part (1) follows immediately from Propositions 2.2, 2.3, 3.1 and 3.2. In Part (2), boundedness of Ω_+ follows from the conservation laws, while continuity follows from the corresponding statements in Propositions 2.1 and 3.1.

□

The solutions of the equation (1.1) constructed in Proposition 3.3, part 1 are dispersive at $+\infty$, but no claim is made at this stage on their behaviour at $-\infty$. Dispersiveness at $-\infty$ would be a consequence of asymptotic completeness, which will be studied only in Section 4.

4 Scattering theory II. Asymptotic completeness.

In this section, we continue the study of the theory of scattering for the Hartree equation (1.1) by addressing the second question raised in the introduction, namely that of asymptotic completeness. In particular we prove the main result of this paper, namely the fact that asympotic completeness holds in the energy space H^1 for radial and suitably repulsive potentials (see Assumption (H3) below). In view of the results of Section 3, especially Proposition 3.2, it will turn out that the crux of the argument consists in showing that the global solutions of the equation (1.1) in $X^1_{loc}(I\!R)$ constructed in Proposition 2.3 actually belong to $X^1(I\!R)$, namely exhibit the time decay properties contained in the definition of that space. The proof uses the method of Morawetz and Strauss [22] and relies on two estimates. The first one is an elementary propagation estimate which for the Hartree (as well as for the NLS) equation replaces the finiteness of the propagation speed for the NLKG equation (see Lemma 4.2). The second estimate follows from the Morawetz inequality, which is closely related to the approximate dilation invariance of the equation (see Proposition 4.1). Space time is split into an internal and an external region where $|x|$ is small or large respectively as compared with $|t|$. For radial repulsive potentials according to the assumption (H3) below, the Morawetz inequality implies an a priori estimate for a suitable norm of the internal part of u (see Proposition 4.2

and Lemma 4.4). One uses that estimate in the internal region and the propagation estimate in the external region. Plugging those estimates into the integral equation for the solution u, one proves successively that a suitable norm of u is small in large intervals (see Lemma 4.5) and tends to zero at infinity (see Lemma 4.6) and that u belongs to $X^1(I\!R)$ (see Proposition 4.3).

We continue to assume $n \geq 3$ as in the rest of this paper. We restrict our attention to positive times. We first state an elementary property of H^1 solutions of the free Schrödinger equation. We recall that $2^\star \equiv 2n/(n-2)$.

Lemma 4.1. *Let $u_0 \in H^1$. Let $2 < r \leq 2^\star$. Then $U(t)u_0$ tends to zero in L^r norm when $|t| \to \infty$.*

Proof. We approximate u_0 in H^1 norm by $u'_0 \in L^{\bar{r}} \cap H^1$. By (2.4), Sobolev inequalities and the unitarity of $U(t)$ in H^1, we estimate

$$\begin{aligned} \| \, U(t) \; u_0 \, \|_r \;\; &\leq \;\; \| \, U(t) \; u'_0 \, \|_r \; + C \, \| \, u_0 - u'_0 \, \|_2^{1-\delta(r)} \; \| \, \nabla(u_0 - u'_0) \, \|_2^{\delta(r)} \\ &\leq \;\; (2\pi|t|)^{-\delta(r)} \;\; \| \, u'_0 \, \|_{\bar{r}} \; + C \, \| \, u_0 - u'_0; H^1 \, \| \end{aligned} \tag{4.1}$$

from which the result follows.

□

We next state the propagation property of finite energy solutions of the equation (1.1) mentioned previously. For any function u of space time and for $t \geq 1$, we define

$$u_{\substack{> \\ <}}(t,x) = u(t,x) \; \chi(|x| \substack{\geq \\ <} \; t \text{ Log } t) \tag{4.2}$$

so that $u = u_> + u_<$. This decomposition corresponds to the splitting of space time mentioned previously. There is nothing magic about the function t Log t. It is chosen so as to tend to infinity faster than t and to ensure the divergence of the integral $\int^\infty dt (t \text{ Log} t)^{-1}$.

Lemma 4.2. *Let V satisfy (H1), let $u \in (\mathcal{C} \cap L^\infty)(I\!R, H^1)$ be a solution of the equation (1.1) and let $u_0 = u(0)$. Then*

(1) For any $R > 0$ and any $t \in I\!R$, u satisfies the estimate

$$\int_{|x|\geq R} dx |u(t,x)|^2 \leq \int dx (1 \wedge R^{-1}|x|)|u_0(x)|^2 + R^{-1}|t| \parallel u \parallel_2 \parallel \nabla u ; L^\infty(I\!R, L^2) \parallel \quad . \tag{4.3}$$

(2) For any r with $2 \leq r < 2^\star$, $\parallel u_>(t) \parallel_r$ tends to zero when $t \to \infty$.

Proof. Part (1). We give only the formal computation, which is easily justified at the available level of regularity. For h a suitably smooth real function, we compute

$$\partial_t < u, hu > = \mathrm{Im} \ < u, \nabla h \cdot \nabla u > \tag{4.4}$$

and therefore

$$< u(t), h\ u(t) > \leq < u_0, h\ u_0 > + \parallel \nabla h \parallel_\infty \parallel u \parallel_2 \int_0^t dt' \parallel \nabla u(t') \parallel_2 \tag{4.5}$$

from which (4.3) follows by taking $h(x) = 1 \wedge R^{-1}|x|$.

Part (2) for $r = 2$ follows from (4.3) with $R = t$ Log t and from the Lebesgue dominated convergence theorem applied to the term containing u_0. The result for general r follows by interpolation between that for $r = 2$ and uniform boundedness in $L^{2^\star}$.

□

The second main ingredient of the proof is the Morawetz inequality which for the Hartree equation can be written as follows.

Proposition 4.1. *Let V satisfy (H1) and let $u \in X^1_{loc}(I\!R) \cap L^\infty(I\!R, H^1)$ be a solution of the equation (1.1). Then for any t_1, $t_2 \in I\!R$, $t_1 \leq t_2$, the following inequality holds*

$$- \int_{t_1}^{t_2} dt \int dx\ \rho(x)\ \widehat{x} \cdot (V \star \nabla \rho) \leq 2 \parallel u \parallel_2 \parallel \nabla u ; L^\infty(I\!R, L^2) \parallel \tag{4.6}$$

where $\widehat{x} = |x|^{-1}x$ and $\rho = |u|^2$.

Proof. There are several proofs of the Morawetz inequality for various equations in the literature. Here we follow the version given in [8] [11]. In order to derive the result at the available level of regularity, we introduce the same regularization as in the proof of energy conservation

in Proposition 2.2. We denote $\varphi \star u = u_\varphi$. Let h be a $\mathcal{C}^4$ function of space with bounded derivatives up to order 4. We compute

$$\partial_t \ \mathrm{Im} < u_\varphi, (\nabla h) \cdot \nabla u_\varphi > = \mathrm{Re} \ < (\Delta h) u_\varphi + 2(\nabla h) \cdot \nabla u_\varphi, -(1/2)\Delta u_\varphi + \varphi \star f(u) > \quad . \tag{4.7}$$

The kinetic part of the RHS (namely the terms not containing f) is treated exactly as for the NLS equation in [11]. The term containing f is rewritten by using the fact that

$$\mathrm{Re} \ < (\Delta h) u_\varphi + 2(\nabla h) \cdot \nabla u_\varphi, f(u_\varphi) > = - \int dx \ \rho_\varphi (\nabla h) \cdot (V \star \nabla \rho_\varphi) \tag{4.8}$$

where $\rho_\varphi = |u_\varphi|^2$, and we obtain

$$\begin{aligned} \partial_t \ \mathrm{Im} < u_\varphi, (\nabla h) \cdot \nabla u_\varphi > &= < \nabla_i \ u_\varphi, (\nabla^2_{ij} h) \nabla_j u_\varphi > -(1/4) < u_\varphi, (\Delta^2 h) u_\varphi > \\ - \int dx \ \rho_\varphi (\nabla h) \cdot (V \star \nabla \rho_\varphi) &+ \ \mathrm{Re} \ < (\Delta h) u_\varphi + 2(\nabla h) \cdot \nabla u_\varphi, \varphi \star f(u) - f(u_\varphi) > \end{aligned} \tag{4.9}$$

where $\nabla^2_{ij} h$ is the matrix of second derivatives of h and summation over the dummy indices i and j is understood.

We next let φ tend to δ. We integrate (4.9) in time in the interval $[t_1, t_2]$ and we take the limit of the time integral of the RHS by using the Lebesgue dominated convergence theorem in the time variable. For that purpose, we need an estimate of the integrand which is uniform in φ and integrable in time. Such an estimate is obvious for the kinetic terms. For the terms containing f, it boils down to

$$| < \nabla u \cdot \nabla h, f(u) > | \leq C \parallel \nabla h \parallel_\infty \parallel V \parallel_p \parallel \nabla u \parallel_2 \parallel u \parallel_s^3 \tag{4.10}$$

with $\delta(s) = n/3p \leq 4/3$, so that the RHS of (4.10) belongs to L^2_{loc} in time.

All terms then tend to the obvious limits and we obtain

$$\begin{aligned} \mathrm{Im} \ < u, (\nabla h) \cdot \nabla u > \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} dt \Big\{ < \nabla_i u, (\nabla^2_{ij} h) \nabla_j u > \\ -(1/4) < u, (\Delta^2 h) u > - \int dx \ \rho (\nabla h) \cdot (V \star \nabla \rho) \Big\}(t) \quad . \end{aligned} \tag{4.11}$$

We next take $h = (x^2 + \sigma^2)^{1/2}$ for some $\sigma > 0$ and we compute

$$\begin{aligned} &\nabla h = h^{-1} x \\ &\nabla^2_{ij} h = h^{-1} \left(\delta_{ij} - h^{-2} \ x_i \ x_j \right) \\ &\Delta^2 h = -(n-1)(n-3) h^{-3} - 6(n-3) \sigma^2 h^{-5} - 15 \sigma^4 h^{-7} \end{aligned}$$

so that $\nabla^2_{ij} h$ is a positive matrix and $\Delta^2 h$ is negative. We then obtain an inequality by dropping the kinetic terms in the RHS of (4.11). Taking in addition the harmless limit $\sigma \to 0$ by the Lebesgue dominated convergence theorem, we obtain

$$- \int_{t_1}^{t_2} dt \int dx\ \rho(x) \widehat{x} \cdot (V \star \nabla \rho) \leq \mathrm{Im} < u, \widehat{x} \cdot \nabla u > \Big|_{t_1}^{t_2} \tag{4.12}$$

from which (4.6) follows immediately.

□

For sufficiently regular V, for instance for $V \in \mathcal{C}^1$ with compact support, the integrand of the time integral in (4.12) can be rewritten as

$$- \int dx\ \rho(x) \widehat{x} \cdot (V \star \nabla \rho) = -(1/2) \int dx\ dy (\widehat{x} - \widehat{y}) \cdot \nabla V(x-y) \rho(x)\ \rho(y) \tag{4.13}$$

which is suggestive of algebraic manipulations to be made below*.

The assumptions on V made so far are not stronger than those made in Section 2. In particular the assumptions on u made in Lemma 4.2 and Proposition 4.1 are ensured by the assumptions of Proposition 2.3, namely (H1) (H2) and $p_2 > n/4$, for any initial data in H^1. In order to proceed further, we need to exploit the fact that the LHS of (4.6) controls some suitable norm of u and for that purpose we need a repulsivity condition on V. That condition takes the following form.

(H3) V is radial and nonincreasing, namely $V(x) = v(|x|)$ where v is nonincreasing in $I\!R^+$. Furthermore, for some $\alpha \geq 2$, v satisfies the following condition.

(Aα) There exists $a > 0$ and $A_\alpha > 0$ such that

$$v(r_1) - v(r_2) \geq \alpha^{-1}\ A_\alpha (r_2^\alpha - r_1^\alpha) \quad \text{for } 0 < r_1 < r_2 \leq a \quad . \tag{4.14}$$

Note that as soon as $V \in L^p$ for some $p < \infty$, (H3) implies that V is nonnegative and tends to zero at infinity, so that (H3) implies (H2) with $V_- = 0$. We next discuss the last condition in (H3).

*The contribution of the Hartree interaction to the Morawetz inequality given in [7] (Theorem 7.4.4) is incorrect. The error appears in (7.4.13) through the application of the radial derivative to a convolution product.

The condition (Aα) can be formulated for any real $\alpha \neq 0$. In all cases it means that v is sufficiently decreasing in $(0, a]$. If $v \in \mathcal{C}^1((0,a])$, it is equivalent to the fact that

$$- v'(r) \geq A_\alpha \; r^{\alpha-1} \qquad \text{for } 0 < r \leq a \quad . \tag{4.15}$$

It is easy to see that for any α, β with $\alpha \neq 0 \neq \beta$ and $\alpha \leq \beta$, (Aα) implies (Aβ) with $A_\beta = a^{\alpha-\beta} A_\alpha$. This is obvious from (4.15) if $v \in \mathcal{C}^1((0,a])$. In the general case, it reduces to the fact that $\alpha^{-1}(r_2^\alpha - r_1^\alpha) \geq \beta^{-1} a^{\alpha-\beta}(r_2^\beta - r_1^\beta)$ for $0 < r_1 \leq r_2 \leq a$ or equivalently, by making the worst choice of a, namely $a = r_2$, and by scaling

$$\alpha^{-1}(1 - r^\alpha) \geq \beta^{-1}(1 - r^\beta) \qquad \text{for } 0 < r < 1 \quad ,$$

which can be verified easily.

¿From the previous discussion it follows that the condition $\alpha \geq 2$ could be dropped in the assumption (H3) without modifying that assumption. We have included that condition because we shall use it in the subsequent applications.

For any real $\alpha \neq 0$, a potential $V(x) = C - \alpha^{-1} A_\alpha |x|^\alpha$ for $0 < |x| \leq a$ satisfies, actually saturates the condition (Aα). The condition (Aα) in (H3) means that V is not too flat at the origin. For instance the potential $V(x) = 1 - \exp(-1/|x|)$ does not satisfy (H3), because there is no α for which it satisfies (Aα).

In order to justify some computations to be made below, we shall need to approximate potentials satisfying (H3) by potentials in $\mathcal{C}^1(I\!\!R^n \setminus \{0\})$ still satisfying that condition. The standard approximation by convolution such as that used in the proof of energy conservation is not suitable for that purpose because it does not preserve in an obvious way the last condition of (H3). We proceed instead as follows. Let $\varphi \in \mathcal{C}^\infty$ be radial nonnegative supported in the unit ball and satisfy $\int dx \; \varphi(x) = 1$. For $V \in L^1_{loc}(I\!\!R^n)$ and $j \geq 2$, we define

$$V_j(x) = j^n |x|^{-n} \int dy \; V(y) \; \varphi\left(j|x|^{-1}(x-y)\right) \tag{4.16}$$

or equivalently

$$V_j(x) = \int dz \; V(x - j^{-1}|x|z) \; \varphi(z) \quad . \tag{4.17}$$

Obviously $V_j \in \mathcal{C}^\infty(I\!\!R^n \setminus \{0\})$. The previous regularisation satisfies the following properties.

Lemma 4.3.

(1) Let $1 \le p \le \infty$ and $V \in L^p$. Then $V_j \in L^p$ and

$$\| V_j \|_p \le (1 - j^{-1})^{-1/p} \ \| V \|_p \quad . \tag{4.18}$$

Let $1 \le p < \infty$. Then V_j tends to V in L^p norm when $j \to \infty$.

(2) If V is radial, then V_j is radial. If in addition V is nonincreasing, then V_j is nonincreasing. If in addition V satisfies the condition $(A\alpha)$ for some $\alpha > 0$, then V_j also satisfies $(A\alpha)$ with a replaced by $a_j = a(1 + j^{-1})^{-1}$ and with A_α replaced by $A_{\alpha j} = A_\alpha (1 - j^{-1})^\alpha$.

Proof. Part (1). The case $p = \infty$ is obvious on (4.17). For $p = 1$, we estimate

$$\| V_j \|_1 \le \int dx \ dz \ |V(y)| \ \varphi(z) = \int dy \ dz |Dy/Dx|^{-1} \ |V(y)| \ \varphi(z)$$

where $y = x - j^{-1}|x|z$ and $|Dy/Dx|$ is the Jacobian of the transformation $x \to y$ for fixed z. One computes

$$|Dy/Dx| = 1 - j^{-1} \widehat{x} \cdot z \ge 1 - j^{-1}$$

which implies (4.18) for $p = 1$. The general case of (4.18) follows by interpolation.

Convergence of V_j to V in L^p for $p < \infty$ follows from the identity

$$V_j(x) - V(x) = \int dz \ \varphi(z) \left(V(x) - V(x - j^{-1}|x|y) \right)$$

for $V \in \mathcal{C}^1$ with compact support and follows from that special case for general V by a density argument.

Part (2). Obviously V radial implies V_j radial. Let now x_1 and x_2 be collinear, $x_1 = |x_1|\widehat{x}$, $x_2 = |x_2|\widehat{x}$ with $0 < |x_1| < |x_2|$. Then

$$V_j(x_1) - V_j(x_2) = \int dz \ \varphi(z) \left(V(x_1 - j^{-1}|x_1|z) - V(x_2 - j^{-1}|x_2|z) \right) \quad .$$

Clearly $x_i - j^{-1}|x_i|z = |x_i|(\widehat{x} - j^{-1}z)$, $i = 1, 2$, are collinear and in the same ratio as x_1 and x_2. Therefore V_j is nonincreasing if V is. Furthermore, if V satisfies (4.14) and if $|x_2|(1 + j^{-1}) \le a$, then

$$\begin{aligned} V_j(x_1) - V_j(x_2) &\ge \alpha^{-1} A_\alpha \left(|x_2|^\alpha - |x_1|^\alpha \right) \int \varphi(z) \ dz \ |\widehat{x} - j^{-1}z|^\alpha \\ &\ge \alpha^{-1} \ A_\alpha (1 - j^{-1})^\alpha \left(|x_2|^\alpha - |x_1|^\alpha \right) \end{aligned}$$

which completes the proof of Part (2).

$\square$

In order to exploit the Morawetz inequality (4.6), we shall need the following spaces. Let $\sigma > 0$ and let Q_i be the cube with edge σ centred at $i\sigma$ where $i \in \mathbb{Z}^n$ so that $I\!R^n = \underset{i}{\cup} Q_i$. Let $1 \leq r,\ m \leq \infty$. We define

$$\ell^m(L^r) = \{u \in L^r_{loc} : \| u; \ell^m(L^r) \| = \| \, \| u; L^q(Q_i) \|; \ell^m \| < \infty\} \quad .$$

The space $\ell^m(L^r)$ does not depend on σ, and different values of σ yield equivalent norms. The previous spaces have been introduced by Birman and Solomjak [4]. They allow for an independent characterization of local regularity and of decay at infinity in terms of integrability properties. The Hölder and Young inequalities hold in those spaces, with the exponents m and r treated independently. See [11] for more details.

We now turn to one of the main points of the proof, namely the fact that the Morawetz inequality allows for the control of the $\ell^m(L^2)$ norm of the internal part of u for some m.

Proposition 4.2. *Let V satisfy (H1) with $n/4 < p_2 \leq p_1 < \infty$ and (H3) and let $u \in X^1_{loc}(I\!R)$ be a solution of the equation (1.1). Then for any $t_1,\ t_2 \in I\!R$ with $1 \leq t_1 \leq t_2$, the following estimate holds*

$$\int_{t_1}^{t_2} dt (t \text{ Log } t + a)^{-1} \ \| u_<(t); \ell^{\alpha+4}(L^2) \|^{\alpha+4} \leq C \ A_\alpha^{-1} \| u \|_2 \sqrt{E} \left(\sqrt{E} + \| u \|_2 \right)^\alpha \tag{4.19}$$

where $u_<$ is defined by (4.2) and where C depends only on n, α and a.

Proof. We first note that under the assumptions made on V, u satisfies the conservation laws of the L^2-norm and of the energy and belongs to $L^\infty(I\!R, H^1)$ with $\| \nabla u; L^\infty(I\!R, L^2) \|^2 \leq 2E$. The crucial steps in the proof can be easily performed if V is $\mathcal{C}^1$ with compact support and we therefore approximate the actual V by potentials of this type. We first approximate V by V_j defined by (4.16) and we then truncate V_j both at the origin and at infinity. Let $\psi_1 \in \mathcal{C}^\infty(I\!R^n)$, ψ_1 even, with $0 \leq \psi_1 \leq 1$, $\psi_1(x) = 1$ for $|x| \leq 1$ and $\psi_1(x) = 0$ for $|x| \geq 2$ and define $\psi_\ell(x) = \psi_1(x/\ell)$ for any $\ell > 0$. The final approximation consists in approximating V by

$V_{jk} \equiv (1-\psi_{1/k})\psi_k\ V_j$ for large j and k. Clearly if $V \in L^p$ with $1 \leq p < \infty$, then V_{jk} tends to V_j in L^p when $k \to \infty$ while V_j tends to V in L^p when $j \to \infty$ by Lemma 4.3 part (1).

We now define

$$J(V) = -\int_{t_1}^{t_2} dt \int dx\ \rho(x)\widehat{x} \cdot (V \star \nabla\rho) \tag{4.20}$$

so that by (4.6)

$$J(V) \leq 2 \parallel u \parallel_2 \sqrt{2E} \quad . \tag{4.21}$$

We define $J(V_j)$ and $J(V_{jk})$ in the same way, for the same ρ. From estimates similar to (4.10) it follows that $J(V)$ is a continuous function of $V \in L^p$. In particular $J(V_j) - J(V) \equiv \varepsilon_j$ tends to zero when $j \to \infty$. Clearly

$$J(V_j) \leq 2 \parallel u \parallel_2 \sqrt{2E} + \varepsilon_j \quad . \tag{4.22}$$

Similarly $J(V_{jk})$ tends to $J(V_j)$ when $k \to \infty$. Using (4.13) and omitting from now on the limits in the time integration, we rewrite $J(V_{jk})$ as

$$\begin{aligned} J(V_{jk}) = &- \int dt \int dx\ \rho\ \widehat{x} \cdot \Big((1-\psi_{1/k})\psi_k\ \nabla V_j \star \rho \Big) \\ &+ \int dt \int dx\ \rho\ \widehat{x} \cdot \Big((\nabla\psi_{1/k} - \nabla\psi_k) V_j \star \rho \Big) \quad . \end{aligned} \tag{4.23}$$

We now estimate

$$\begin{aligned} \left| \int dx\ \rho\ \widehat{x} \cdot \Big((\nabla\psi_{1/k})\ V_j \star \rho \Big) \right| &\leq C \parallel V_j \parallel_p \parallel u \parallel_s^4 \parallel \nabla\psi_{1/k} \parallel_\ell \\ &\leq C\ k^{1-n/\ell} \ \parallel V_j \parallel_p \parallel u \parallel_s^4 \end{aligned} \tag{4.24}$$

with $4\delta(s) = n/p + n/\ell$. We choose $5/4 < \delta(s) \leq 3/2$ which for $n/p \leq 4$ implies $n/\ell > 1$ and also ensures the local time integrability of the RHS of (4.24), so that the contribution of the LHS thereof to (4.23) tends to zero when $k \to \infty$. The contribution of $\nabla\psi_k$ to (4.23) is treated in the same way. We obtain therefore

$$J(V_j) = \lim_{k\to\infty} - \int dt \int dx\ \rho\ \widehat{x} \cdot \Big((1-\psi_{1/k})\psi_k\ \nabla V_j \star \rho \Big) \quad . \tag{4.25}$$

We now use the fact that V_j is radial with $V_j(x) \equiv v_j(|x|)$ and that ψ_1 is even to rewrite (4.25) as

$$\begin{aligned} J(V_j) = \lim_{k\to\infty} -\frac{1}{2} \int dt \int dx\ dy\ \rho(x)\ \rho(y) (\widehat{x} - \widehat{y}) \cdot (x-y)|x-y|^{-1} \\ v_j'(|x-y|) \Big((1-\psi_{1/k})\psi_k \Big) (x-y) \quad . \end{aligned} \tag{4.26}$$

Now $v_j' \le 0$ by (H3) and Lemma 4.3 part (2), while

$$(\widehat{x} - \widehat{y}) \cdot (x - y) = \Big(|x| \ |y| - x \cdot y)(|x|^{-1} + |y|^{-1} \Big) \ge 0 \tag{4.27}$$

so that the integrand in (4.26) is nonnegative and therefore nondecreasing in k. Therefore, by the monotone convergence theorem

$$J(V_j) = -(1/2) \int dt \int dx \ dy \ \rho(x) \ \rho(y) (\widehat{x} - \widehat{y}) \cdot (x - y) |x - y|^{-1} v_j'(|x - y|) \quad . \tag{4.28}$$

We now use the assumption (H3) applied to V_j according to Lemma 4.3 part (2) and the estimate (4.22) to conclude that

$$\int dt \int_{|x-y| \le a_j} dx \ dy \ \rho(x) \ \rho(y) (\widehat{x} - \widehat{y}) \cdot (x - y) |x - y|^{\alpha - 2} \le 2 A_{\alpha j}^{-1} (2 \parallel u \parallel_2 \sqrt{2E} + \varepsilon_j) \quad .$$

Taking the limit $j \to \infty$ yields

$$\int dt \int_{|x-y| \le a} dx \ dy \ \rho(x) \ \rho(y) (\widehat{x} - \widehat{y}) \cdot (x - y) |x - y|^{\alpha - 2} \le 4 A_\alpha^{-1} \parallel u \parallel_2 \sqrt{2E} \quad . \tag{4.29}$$

The next step in the proof consists in showing that the LHS of (4.29) controls a suitable $\ell^m(L^2)$ norm of $u_<$. Let $y_\parallel$ and $y_\perp$ be the components of y parallel and normal to x. Then

$$\frac{|x| \ |y| - x \cdot y}{|x|} = \frac{|x| y_\perp^2}{|x| \ |y| + x \cdot y} \ge \frac{y_\perp^2}{2|y|} \quad . \tag{4.30}$$

Substituting (4.27) and (4.30) into (4.29) and using the fact that $|y| \le |x| + a$ and $|y_\perp| \le |x - y|$, we obtain

$$\int dt \int dx \ \rho(x) (|x| + a)^{-1} \int_{|x-y| \le a} dy \ \rho(y) |y_\perp|^\alpha \le 4 A_\alpha^{-1} \parallel u \parallel_2 \ \sqrt{2E} \quad . \tag{4.31}$$

We now derive a lower bound of the integral over y for fixed x. We first restrict that integral to the cylinder C_x of center x and axis x with diameter and height $a\sqrt{2}$. For fixed $y_\parallel$, we consider the integral over $y_\perp$ which takes place in the ball B of radius $a/\sqrt{2} \equiv a_1$ centered at the origin in $I\!R^{n-1}$. Let $r_\perp = |y_\perp|$ and let w be the vector field in $I\!R^{n-1}$

$$w = y_\perp \, (a_1^\alpha - r_\perp^\alpha)$$

so that

$$\nabla \cdot w = (n-1) a_1^\alpha - (n - 1 + \alpha) r_\perp^\alpha \quad .$$

We now write

$$\int_B dy_\perp \ \rho \ \nabla \cdot w = - \int_B dy_\perp \ w \cdot \nabla \rho$$

so that

$$\begin{aligned}(n-1) a_1^\alpha \int_B dy_\perp \ \rho &= (n-1+\alpha) \int_B dy_\perp \ r_\perp^\alpha \ \rho \ - \ \int_B dy_\perp (a_1^\alpha - r_\perp^\alpha) y_\perp \cdot \nabla \rho \\ &\leq (n-1+\alpha) \int_B dy_\perp \ r_\perp^\alpha \ \rho \ + \ 2a_1^\alpha \parallel r_\perp u; L^2(B) \parallel \parallel \nabla u; L^2(B) \parallel \quad . \end{aligned} \tag{4.32}$$

We next integrate (4.32) over $y_\parallel$ and estimate the second term in the RHS by applying the Schwarz inequality and extending the integral of ∇u to the whole of $I\!R^n$. We obtain

$$(n-1) a_1^\alpha \parallel u; L^2(C_x) \parallel^2 \leq (n-1+\alpha) \parallel r_\perp^{\alpha/2} u; L^2(C_x) \parallel^2 \ + 2a_1^\alpha \parallel \nabla u \parallel_2 \parallel r_\perp u; L^2(C_x) \parallel$$

$$\leq (n-1+\alpha) \parallel r_\perp^{\alpha/2} u; L^2(C_x) \parallel^2 \ + 2a_1^\alpha \parallel \nabla u \parallel_2 \parallel r_\perp^{\alpha/2} u; L^2(C_x) \parallel^{2/\alpha} \parallel u; L^2(C_x) \parallel^{1-2/\alpha}$$

by the Hölder inequality, and therefore

$$(n-1) \parallel u; L^2(C_x) \parallel^{1+2/\alpha} \leq \parallel r_\perp^{\alpha/2} u; L^2(C_x) \parallel^{2/\alpha} \left\{ 2 \parallel \nabla u \parallel_2 + (n-1+\alpha) a_1^{-1} \parallel u \parallel_2 \right\} \quad .$$

Finally

$$\int_{C_x} dy \ r_\perp^\alpha \ \rho(y) = \parallel r_\perp^{\alpha/2} u; L^2(C_x) \parallel^2 \geq M \parallel u; L^2(C_x) \parallel^{\alpha+2} \tag{4.33}$$

with

$$M = (n-1)^\alpha \left\{ 2\sqrt{2E} + (n-1+\alpha)\sqrt{2} a^{-1} \parallel u \parallel_2 \right\}^{-\alpha} \quad . \tag{4.34}$$

Substituting (4.33) into (4.31) yields

$$M \int dt \int dx \ \rho(x)(|x|+a)^{-1} \parallel u; L^2(C_x) \parallel^{\alpha+2} \leq 4 A_\alpha^{-1} \parallel u \parallel_2 \sqrt{2E} \quad . \tag{4.35}$$

We obtain a lower bound of the LHS of (4.35) by replacing u by $u_<$ and $(|x|+a)^{-1}$ by $(t \ \mathrm{Log} \ t + a)^{-1}$ according to (4.2). Introducing in addition the decomposition of $I\!R^n$ in unit cubes appropriate to the definition of $\ell^m(L^2)$ spaces, we obtain

$$M \int dt (t \ \mathrm{Log} \ t + a)^{-1} \sum_i \int_{Q_i} dx |u_<(x)|^2 \parallel u_<; L^2(C_x) \parallel^{\alpha+2} \leq 4 A_\alpha^{-1} \parallel u \parallel_2 \sqrt{2E} \quad . \tag{4.36}$$

We next choose σ in such a way that $C_x \supset Q_i$ for all $x \in Q_i$, and for that purpose we take $\sigma = a(2n)^{-1/2}$, and we obtain

$$M \int dt (t \ \mathrm{Log} \ t + a)^{-1} \ \parallel u_<; \ell^{\alpha+4}(L^2) \parallel^{\alpha+4} \leq 4 A_\alpha^{-1} \parallel u \parallel_2 \sqrt{2E} \quad . \tag{4.37}$$

The estimate (4.19) now follows from (4.37) and (4.34).

□

Remark 4.1. If the potential V is flat at the origin, it seems difficult to extract a norm estimate of u directly from the inequality (4.6). In fact if V is $\mathcal{C}^1$ with compact support, radial and nonincreasing with Supp $v' \subset \{r : 0 < a \leq r \leq b\}$ and if

$$\text{Supp } u \subset \bigcup_{i \in \mathbb{Z}^n} B((a+b)i, a/2) \tag{4.38}$$

where $B(x, r)$ is the ball of center x and radius r, then the RHS of (4.13) is zero. Therefore in order to get some information from (4.6), one would have to use again properties of the evolution, for instance the fact that the support property (4.38) cannot be preserved in time.

The basic estimate (4.19) is not convenient for direct application to the integral equation, and we now derive a more readily usable consequence thereof (cf. Lemma 5 in [20] and Lemma 5.3 in [11]).

Lemma 4.4. *Let V satisfy (H1) with $n/4 < p_2 \leq p_1 < \infty$ and (H3) and let $u \in X^1_{loc}(I\!R)$ be a solution of the equation (1.1). Then, for any $t_1 \geq 1$, for any $\varepsilon > 0$ and for any $\ell \geq a$, there exists $t_2 \geq t_1 + \ell$ such that*

$$\int_{t_2-\ell}^{t_2} dt \parallel u_<(t); \ell^{\alpha+4}(L^2) \parallel^{\alpha+4} \leq \varepsilon \quad . \tag{4.39}$$

One can find such a t_2 satisfying

$$t_2 \leq \exp\left\{(1 + \text{Log}(t_1 + \ell)) \exp(M\ell/\varepsilon) - 1\right\} \tag{4.40}$$

where M is the RHS of (4.19), namely

$$M = C\ A_\alpha^{-1} \parallel u \parallel_2\ \sqrt{E}\left(\sqrt{E} + \parallel u \parallel_2\right)^\alpha \quad . \tag{4.41}$$

Proof. Let N be a positive integer. From (4.19) we obtain

$$M \geq \sum_{j=1}^{N} \left[(t_1 + j\ell)\ \text{Log}(t_1 + j\ell) + a\right]^{-1}\ K_j$$

where

$$K_j = \int_{t_1+(j-1)\ell}^{t_1+j\ell} dt \parallel u_<(t); \ell^{\alpha+4}(L^2) \parallel^{\alpha+4} \quad .$$

If $K_j \geq \varepsilon$ for $1 \leq j \leq N$, then

$$\begin{aligned} M &\geq \varepsilon \sum_{j=1}^{N} [(t_1+j\ell) \ \text{Log} \ (t_1+j\ell)+a]^{-1} \\ &\geq \varepsilon \ \ell^{-1} \int_{t_1+\ell}^{t_1+(N+1)\ell} dt \, (t \ \text{Log} \ t + a)^{-1} \\ &\geq \varepsilon \ \ell^{-1} \ \text{Log} \ \left\{ (1+\text{Log}\,(t_1+(N+1)\ell)) \, (1+\text{Log}\,(t_1+\ell))^{-1} \right\} \end{aligned}$$

which is an upper bound on N, namely

$$t_1 + (N+1)\ell \leq \exp\{(1+\text{Log}\,(t_1+\ell))\exp(M\ell/\varepsilon) - 1\} \quad .$$

For the first N not satisfying that estimate, there is a j with $K_j \leq \varepsilon$ and one can take $t_2 = t_1 + j\ell$ for that j. That t_2 is easily seen to satisfy (4.40).

$\square$

We now exploit the estimates of Lemmas 4.2 and 4.4 together with the integral equation for u to prove that for $2 < r < 2^\star$, the L^r norm of u is small in large intervals. That part of the proof follows the version given in [7] [8] for the NLS equation.

The assumptions made so far on V include only (H1) with $n/4 < p_2 \leq p_1 < \infty$ and the repulsivity condition (H3), but do not include any decay property at infinity. In order to proceed further, we shall now require such properties, in the form of upper bounds on p_1. In particular we require $p_1 < n$ for the next result (and subsequently $p_1 < n/2$).

Lemma 4.5. *Let V satisfy (H1) with $1 \vee \ n/4 < p_2 \leq p_1 < n$ and (H3), and let $u \in X^1_{loc}(I\!R)$ be a solution of the equation (1.1). Let $2 < r < 2^\star$. Then for any $\varepsilon > 0$ and for any $\ell_1 > 0$, there exists $t_2 \geq \ell_1$ such that*

$$\parallel u; L^\infty\left([t_2 - \ell_1, t_2]\,, L^r\right) \parallel \leq \varepsilon \quad . \tag{4.42}$$

Proof. Since $u \in L^\infty(I\!R, H^1)$, it is sufficient to derive the result for one value of r with $2 < r < 2^\star$. The result for general r will then follow by interpolation with boundedness in H^1.

We take one such r, with $0 < \delta \equiv \delta(r) < 1$. Various compatible conditions will be imposed on r in the course of the proof. For technical reasons, we also introduce an $r_1 > 2^\star$, namely with $\delta_1 \equiv \delta(r_1) > 1$, which will also have to satisfy various compatible conditions.

For future reference, we note that for any s_1, s_2, $t \in I\!R$

$$G(s_1, s_2, t) = -i \int_{s_1}^{s_2} dt' \ U(t-t') \ f(u(t')) = U(t-s_2) \ u(s_2) - U(t-s_1) \ u(s_1)$$

so that

$$\| \ G(s_1, s_2, t) \ \|_2 \leq 2 \ \ \| \ u \ \|_2 \quad . \tag{4.43}$$

Let now ε and ℓ_1 be given. We introduce $\ell_2 \geq 1$, $t_1 > 0$ and $t_2 \geq t_1 + \ell$ where $\ell = \ell_1 + \ell_2$, to be chosen later ; ℓ_2 and t_1 will have to be sufficiently large, depending on ε but not on ℓ_1 for given u. We write the integral equation for u with $t \in [t_2 - \ell_1, t_2]$ and we split the integral in that equation as follows

$$\begin{aligned} u(t) \quad &= \quad U(t) \ u(0) - i \left\{ \int_0^{t-\ell_2} dt' + \int_{t-\ell_2}^{t-1} dt' + \int_{t-1}^{t} dt' \right\} U(t-t') \ f(u(t')) \\ &\equiv \quad u^{(0)} + u_1 + u_2 + u_3 \end{aligned} \tag{4.44}$$

in obvious notation, and we estimate the various terms in L^r successively. Those estimates will require auxiliary estimates of $f(u)$ in various spaces, and the latter will be postponed to the end of the proof. In all the proof, M denotes various constants depending only on r, r_1 and $\| \ u \ \|_2$, $E(u)$, possibly varying from one estimate to the next.

Estimate of $\mathbf{u^{(0)}}$.

It follows from Lemma 4.1 that

$$\varepsilon_0(t) \ \equiv \| \ U(t) \ u(0); L^\infty([t, \infty), L^r) \ \| \to 0 \tag{4.45}$$

when $t \to \infty$, so that for $t \geq t_2 - \ell_1$

$$\| \ u^{(0)}(t) \ \|_r \leq \varepsilon_0(t) \leq \varepsilon_0 \left(t_2 - \ell_1 \right) \leq \varepsilon_0(\ell_2) \leq \varepsilon/4 \tag{4.46}$$

for ℓ_2 sufficiently large depending on ε.

Estimate of $\mathbf{u_1}$.

We estimate by the Hölder inequality

$$\| \ u_1 \ \|_r \leq \| \ u_1 \ \|_2^{1-\delta/\delta_1} \ \| \ u_1 \ \|_{r_1}^{\delta/\delta_1} \quad . \tag{4.47}$$

The L^2 norm of u_1 is estimated by (4.43). The L^{r_1} norm is estimated by the use of the pointwise estimate (2.4) as

$$\begin{aligned} \| u_1 \|_{r_1} &\leq C \int_0^{t-\ell_2} (t-t')^{-\delta_1} \ \| f(u(t')) \|_{\bar{r}_1} \\ &\leq C \left(\delta_1 - 1\right)^{-1} \ \ell_2^{1-\delta_1} \ \| f(u); L^\infty(I\!R, L^{\bar{r}_1}) \| \quad . \end{aligned} \tag{4.48}$$

We now use the estimate

$$\| f(u); L^\infty(I\!R, L^{\bar{r}_1}) \| \leq M \tag{4.49}$$

the proof of which is postponed. By (4.47) (4.48) (4.49) we can ensure that for $t_2 - \ell_1 \leq t \leq t_2$

$$\| u_1(t) \|_r \leq M \ \ell_2^{\delta/\delta_1 - \delta} \leq \varepsilon/4 \tag{4.50}$$

for ℓ_2 sufficiently large depending on ε. We now choose $\ell_2 = \ell_2(\varepsilon)$ so as to ensure both (4.46) and (4.50).

We now turn to the estimate of u_2 and u_3. Here we need to consider the contributions of the internal and external regions separately. We define

$$f_{\stackrel{<}{>}}(u) = u \left(V \star |u_{\stackrel{<}{>}}|^2 \right) \tag{4.51}$$

and for $i = 2, 3$

$$u_i^{\stackrel{<}{>}}(t) = -i \int dt' \ U(t-t') \ f_{\stackrel{<}{>}}(u(t')) \tag{4.52}$$

where the time integral is performed in the appropriate interval, so that $f(u) = f_>(u) + f_<(u)$ and therefore $u_i = u_i^> + u_i^<$. Note that this decomposition is not that defined by (4.2).

Estimate of $\mathbf{u_2}$ and $\mathbf{u_3}$.

We estimate again by the Hölder inequality and (4.43)

$$\begin{aligned} \| u_2 \|_r &\leq \| u_2 \|_2^{1-\delta/\delta_1} \ \| u_2 \|_{r_1}^{\delta/\delta_1} \\ &\leq 2 \, \| u \|_2^{1-\delta/\delta_1} \left(\| u_2^> \|_{r_1}^{\delta/\delta_1} \ + \ \| u_2^< \|_{r_1}^{\delta/\delta_1} \text{ffff} \right) \end{aligned} \tag{4.53}$$

and more simply

$$\| u_3 \|_r \leq \| u_3^> \|_r \ + \ \| u_3^< \|_r \quad . \tag{4.54}$$

Contribution of the external region.

By the same computation as in (4.48), we estimate

$$\| u_2^>(t) \|_{r_1} \leq C\,(\delta_1 - 1)^{-1} \;\; \| f_>(u); L^\infty\left([t-\ell_2, t]\,, L^{\bar{r}_1}\right) \| \tag{4.55}$$

and similarly

$$\| u_3^>(t) \|_r \leq C(1-\delta)^{-1} \;\; \| f_>(u); L^\infty\left([t-\ell_2, t]\,, L^{\bar{r}}\right) \| \quad . \tag{4.56}$$

We next use the estimate

$$\| f_>(u(t)); L^{\bar{r}_1} \cap L^{\bar{r}} \| \leq M \| u_>(t) \|_2^\mu \tag{4.57}$$

valid for some $\mu > 0$ and all $t \geq 1$, the proof of which is postponed. Using (4.55)-(4.57) and the propagation result of Lemma 4.2, part (2), we can ensure that the contribution of the external region to $\| u_2(t) \|_r + \| u_3(t) \|_r$ which can be read on (4.53) (4.54) satisfies

$$2 \| u \|_2^{1-\delta/\delta_1} \; \| u_2^>(t) \|_{r_1}^{\delta/\delta_1} + \| u_3^>(t) \|_r \leq M \| u_>; L^\infty\left([t_2 - \ell_1 - \ell_2, t_2]\,, L^2\right) \|^{\mu\delta/\delta_1} \leq \varepsilon/4 \tag{4.58}$$

for all $t \in [t_2 - \ell_1, t_2]$ by taking t_1 sufficiently large, depending on ε, since we have imposed $t_2 \geq t_1 + \ell_1 + \ell_2$. We now choose $t_1 = t_1(\varepsilon)$ such that (4.58) holds.

Contribution of the internal region.

We shall use the estimate

$$\| f_<(u(t)); L^{\bar{r}_1} \cap L^{\bar{r}} \| \leq M \| u_<(t); \ell^m(L^2) \|^{m/s} \tag{4.59}$$

where $m = \alpha + 4$, valid for some s with $0 < s^{-1} < 1 - \delta$ and for all $t \geq 1$, the proof of which is postponed. Using again the pointwise estimate (2.4), we estimate

$$\begin{aligned} \| u_2^<(t) \|_{r_1} &\leq C \int_{t-\ell_2}^{t-1} dt'(t-t')^{-\delta_1} \;\; \| f_<(u(t')) \|_{\bar{r}_1} \\ &\leq M(\delta_1 \bar{s} - 1)^{-1} \left\{ \int_{t-\ell_2}^{t-1} dt' \;\; \| u_<(t'); \ell^m(L^2) \|^m \right\}^{1/s} \end{aligned} \tag{4.60}$$

by the Hölder inequality in time and (4.59), and similarly

$$\begin{aligned} \| u_3^<(t) \|_r &\leq C \int_{t-1}^{t} dt'(t-t')^{-\delta} \;\; \| f_<(u(t')) \|_{\bar{r}} \\ &\leq M(1 - \delta\bar{s})^{-1} \left\{ \int_{t-1}^{t} dt' \;\; \| u_<(t'); \ell^m(L^2) \|^m \right\}^{1/s} \quad . \end{aligned} \tag{4.61}$$

We now use (4.60) (4.61) and we apply Lemma 4.4 to conclude that there exists $t_2 \geq t_1 + \ell \equiv t_1 + \ell_1 + \ell_2$ (remember that ℓ_2 and t_1 are already chosen, depending on ε) such that the contribution of the internal region to $\| u_2(t) \|_r + \| u_3(t) \|_r$, which can be read on (4.53) (4.54), satisfies

$$2 \| u \|_2^{1-\delta/\delta_1} \| u_2^<(t) \|_{r_1}^{\delta/\delta_1} + \| u_3^<(t) \|_r \leq M \left\{ \int_{t_2-\ell}^{t_2} dt' \ \| u_<(t'); \ell^m(L^2) \|^m \right\}^{\delta/s\delta_1} \leq \varepsilon/4 \quad (4.62)$$

for all $t \in [t_2 - \ell_1, t_2]$.

Collecting (4.46) (4.50) (4.58) and (4.62) and comparing with (4.44) (4.53) and (4.54) yields (4.42).

It remains to prove the estimates (4.49) (4.57) and (4.59) on f. Estimates of a quantity involving $\bar{r}_{(1)}$ mean that we want the estimates both for $\bar{r}$ and for $\bar{r}_1$.

Proof of (4.49) and (4.57).

We consider only (4.57), of which (4.49) is the special case obtained by replacing $u_>$ by u and taking $\mu = 0$. We estimate

$$\begin{aligned} \| f_>(u) \|_{\bar{r}_{(1)}} &\leq C \| V \|_p \| u \|_{r_2} \text{ ff } \| u_> \|_{r_3}^2 \\ &\leq C \| V \|_p \| u \|_{r_2} \| u_> \|_2^{\mu} \| u \|_{2^*}^{2-\mu} \end{aligned} \quad (4.63)$$

with $\delta_i = \delta(r_i)$, $i = 2, 3$, $0 \leq \delta_2, \delta_3 \leq 1$,

$$\delta_{(1)} + \delta_2 + 2\delta_3 = n/p$$

and $\mu = 2(1 - \delta_3)$. One can find admissible r_2 and r_3 provided $\delta_1 \leq n/p \leq 3 + \delta - \mu$ which allows for $\delta < 1 < \delta_1$ and $\mu > 0$ provided $1 < n/p < 4$.

Proof of (4.59).

For radial nonincreasing V satisfying (H1), one can decompose V as $V = V_1 + V_2$ where $V_1 \in \ell^{p_1}(L^\infty)$ and $V_2 \in \ell^1(L^{p_2})$. One can take for instance $V_1(x) = V(x) \ \chi(|x| \geq a)$ and $V_2(x) = V(x) \ \chi(|x| \leq a)$. Correspondingly, we decompose $f_<(u) = f_{1<}(u) + f_{2<}(u)$. Using the fact that the spaces $\ell^k(L^r)$ are monotonically increasing in k and decreasing in r, we estimate

$$\begin{aligned} &\| f_{1<}(u) \|_{\bar{r}_{(1)}} \leq C \| f_{1<}(u); \ell^{\bar{r}_1}(L^2) \| \\ &\leq C \| V_1; \ell^{p_1}(L^\infty) \| \| u \|_2 \| u_<; \ell^k(L^2) \|^2 \end{aligned} \quad (4.64)$$

provided

$$n/p_1 \geq \delta_{(1)} + 2\delta(k) \quad .$$

We then estimate

$$\parallel u_<; \ell^k(L^2) \parallel \leq C \parallel u \parallel_2^{1-\lambda} \parallel u_<; \ell^m(L^2) \parallel^\lambda \tag{4.65}$$

for any λ with

$$0 \leq \lambda \leq 1 \wedge \delta(k)/\delta(m)$$

so that $f_{1<}$ satisfies (4.59) provided one can take $\delta(k) > 0$, namely provided $n/p_1 > \delta_1$.

We next estimate $f_{2<}$ as

$$\parallel f_{2<}(u) \parallel_{\bar{r}_1} \leq C \parallel V_2; \ell^1(L^{p_2}) \parallel \parallel u \parallel_{r_2} \parallel u_<; \ell^q(L^{r_3}) \parallel^2 \tag{4.66}$$

with $0 \leq \delta_2$, $\delta_3 \leq 1$,

$$\delta_{(1)} + \delta_2 + 2\delta_3 = n/p_2 \tag{4.67}$$

$$\delta_{(1)} + \delta_2 + 2\delta(q) = n \quad . \tag{4.68}$$

If δ_2 and δ_3 satisfy (4.67), and if $p_2 > 1$ one can use (4.68) to define q satisfying $r_3 < q < \infty$. If in addition $\delta_3 < 1$, one then estimates

$$\parallel u_<; \ell^q(L^{r_3}) \parallel \leq C \parallel u; \ell^k(L^2) \parallel^{1-\delta_3} \parallel u \parallel_{2^*}^{\delta_3} \tag{4.69}$$

with $\delta(k) = (1 - \delta_3)^{-1}(\delta(q) - \delta_3)$, which together with (4.65) (4.66) proves (4.59) for $f_{2<}$.

It remains only to ensure (4.67) with $\delta_3 < 1$. For that purpose we choose $\delta_{(1)} + \delta_2 = n/p_2$ and $\delta_3 = 0$ if $\delta_{(1)} \leq n/p_2 \leq \delta_{(1)} + 1$, and $\delta_{(1)} + 1 + 2\delta_3 = n/p_2$ if $\delta_{(1)} + 1 \leq n/p_2$, which ensures $\delta_3 < 1$ provided $n/p_2 < 3 + \delta$.

Finally the required estimates (4.49) (4.57) and (4.59) hold provided

$$\delta_1 < n/p_1 \leq n/p_2 < n \wedge (3 + \delta)$$

which can be ensured under the assumptions made on V by taking δ and δ_1 sufficiently close to 1.

$\square$

The next step in the proof consists in showing that the L^r norm of $u(t)$ tends to zero when $t \to \infty$. For that purpose we need to reinforce the decay assumption on V at infinity to its final form, namely $p_1 < n/2$.

Lemma 4.6. *Let V satisfy (H1) with $1 \vee n/4 < p_2 \leq p_1 < n/2$ and (H3) and let $u \in X^1_{loc}(I\!R)$ be a solution of the equation (1.1). Let $2 < r < 2^\star$. Then $\| u(t) \|_r$ tends to zero when $t \to \infty$.*

Proof. The main step of the proof consists in showing that if u satisfies (4.42) for some $\varepsilon > 0$ sufficiently small (depending on $\| u \|_2$ and E) and for some ℓ_1 sufficiently large (depending on u and on ε), then there exists b, $0 < b \leq 1$, depending on ε but not on ℓ_1, such that

$$\| u; L^\infty([t_2 - \ell_1, t_2 + b], L^r) \| \leq \varepsilon \quad . \tag{4.70}$$

For that purpose, we write the integral equation for u with $t \in [t_2, t_2 + 1]$ and we split the integral in that equation as follows

$$\begin{aligned} u(t) &= U(t)\ u(0) - i \left\{ \int_0^{t_2-\ell_1} dt' + \int_{t_2-\ell_1}^{t-1} dt' + \int_{t-1}^{t_2} dt' + \int_{t_2}^{t} dt' \right\} U(t-t')\ f(u(t')) \\ &\equiv u^{(0)} + u_1 + u_2 + u_3 + u_4 \end{aligned} \tag{4.71}$$

and we estimate the various terms successively. As in the proof of Lemma 4.5, auxiliary estimates on $f(u)$ are postponed to the end.

Estimate of $\mathbf{u^{(0)}}$.

In the same way as in (4.46), we can ensure that for $t \geq t_2$

$$\| u^{(0)}(t) \|_r \leq \varepsilon_0(t) \leq \varepsilon_0(t_2) \leq \varepsilon_0(\ell_1) \leq \varepsilon/4 \tag{4.72}$$

for ℓ_1 sufficiently large depending on u and on ε.

Estimate of $\mathbf{u_1}$.

By the same estimates as in the proof of Lemma 4.5 (see especially (4.47) (4.48) with ℓ_2 replaced by $\ell_1 + t - t_2$, we can ensure that for $t \geq t_2$

$$\| u_1(t) \|_r \leq M\ \ell_1^{\delta/\delta_1 - \delta} \leq \varepsilon/4 \tag{4.73}$$

for ℓ_1 sufficiently large depending on ε, under the condition (4.49).

We require ℓ_1 to be sufficiently large, depending on ε, to ensure (4.72) and (4.73).

Estimate of $\mathbf{u_4}$.

By the pointwise estimate (2.4), we estimate

$$\begin{aligned} \| u_4(t) \|_r &\leq C \int_{t_2}^{t} dt'(t-t')^{-\delta} \ \| f(u(t')) \|_{\bar{r}} \\ &\leq C(1-\delta)^{-1}(t-t_2)^{1-\delta} \ \| f(u); L^\infty(I\!R, L^{\bar{r}}) \| \end{aligned} \tag{4.74}$$

which by an estimate similar to (4.49) with $\bar{r}_1$ replaced by $\bar{r}$, the proof of which is omitted, enables us to ensure that

$$\| u_4(t) \|_r \leq M \ b^{1-\delta} \leq \varepsilon/4 \tag{4.75}$$

for $t_2 \leq t \leq t_2 + b$ by taking b sufficiently small depending on ε.

Note that the estimates made so far require only $p_1 < n$, but not the stronger condition $p_1 < n/2$. The latter will be essential to estimate u_2 and u_3.

Estimate of $\mathbf{u_2}$ and $\mathbf{u_3}$.

By the same method as in the proof of Lemma 4.5, especially (4.53) (4.55) (4.56) we estimate

$$\| u_2(t) \|_r \leq 2 \| u \|_2^{1-\delta/\delta_1} \ \| u_2(t) \|_{r_1}^{\delta/\delta_1} \tag{4.76}$$

$$\| u_2(t) \|_{r_1} \leq C \left(\delta_1 - 1\right)^{-1} \ \| f(u); L^\infty\left(\left[t_2-\ell_1, t_2\right], L^{\bar{r}_1}\right) \| \tag{4.77}$$

$$\| u_3(t) \|_r \leq C(1-\delta)^{-1} \| f(u); L^\infty\left(\left[t_2-\ell_1, t_2\right], L^{\bar{r}}\right) \| \quad . \tag{4.78}$$

We now use the estimates

$$\| f(u) \|_{\bar{r}_1}^{\delta/\delta_1} \leq M \| u \|_r^{1+\nu} \tag{4.79}$$

$$\| f(u) \|_{\bar{r}} \leq M \| u \|_r^{1+\nu} \tag{4.80}$$

valid for some $\nu > 0$, the proof of which is postponed.

Using (4.76)-(4.80) and the assumption (4.42) made on u, we can ensure that

$$\begin{aligned} \| u_2(t) \|_r \ + \ \| u_3(t) \|_r &\leq M \| u; L^\infty\left(\left[t_2-\ell_1, t_2\right], L^r\right) \|^{1+\nu} \\ &\leq M \ \varepsilon^{1+\nu} \leq \varepsilon/4 \end{aligned} \tag{4.81}$$

for all $t \in [t_2, t_2+1]$, for ε sufficiently small depending only on $\| u \|_2$ and E, namely for $\varepsilon \leq (4M)^{-1/\nu}$.

The proof of Lemma 4.6 now runs as follows. We pick an ε satisfying the previous condition. We next choose ℓ_1 depending on u and ε so as to ensure (4.72) (4.73) for any $t_2 \geq \ell_1$, and we choose b depending on u and ε so as to ensure (4.75). We next apply Lemma 4.5 with the previous ε and ℓ_1, thereby obtaining t_2 such that (4.42) holds. It then follows from (4.72), (4.73), (4.75) and (4.81) that also (4.70) holds. One then iterates the argument with ℓ_1 and t_2 replaced by $\ell_1 + jb$ and $t_2 + jb$ for $j = 1, 2, \cdots$, which is possible since ℓ_1 increases and b is independent on ℓ_1. One then obtains

$$\| u(t); L^\infty([t_2 - \ell_1, \infty), L^r) \| \leq \varepsilon \quad .$$

Applying that result for arbitrarily small ε proves the Lemma.

It remains to prove the estimates (4.79) (4.80) for $f(u)$. For that purpose we estimate

$$\| f(u) \|_{\bar{r}_{(1)}} \leq C \| V \|_p \| u \|_{r_2}^3 \tag{4.82}$$

where

$$\delta_{(1)} + 3\delta_2 = n/p \quad .$$

Assuming without loss of generality that $n/p \leq 4\delta$, we obtain $\delta_2 \leq \delta$ in all cases, and we continue (4.82) as

$$\cdots \leq C \| V \|_p \| u \|_2^{3(1-\delta_2/\delta)} \| u \|_r^{3\delta_2/\delta} \tag{4.83}$$

so that for both (4.79) and (4.80), the condition $\nu > 0$ becomes

$$1 + \nu = 3\delta_2\ \delta_{(1)}^{-1} = \left(n/p - \delta_{(1)}\right) \delta_{(1)}^{-1} > 1$$

or equivalently $n/p > 2\delta_{(1)}$. Therefore (4.79) (4.80) hold provided $2\delta_1 < n/p \leq 4\delta$, which can be ensured under the assumptions made on V by taking δ and δ_1 sufficiently close to 1.

□

We can now state the main result of this paper.

Proposition 4.3. *Let V satisfy (H1) with $1 \vee n/4 < p_2 \leq p_1 < n/2$ and (H3).*

(1) Let $u \in X^1_{loc}(I\!R)$ be a solution of the equation (1.1). Then $u \in X^1(I\!R)$.

(2) The wave operators $\Omega_\pm$ are bijective bounded and continuous and their inverses $\Omega_\pm^{-1}$ are bounded and continuous from H^1 to H^1.

Proof. Part (1). We give the proof in the special case where $V \in L^p$. The general case of V satisfying (H1) with $p_2 < p_1$ can be treated by a straightforward extension of that proof, based on the fact that admissible pairs (q_1, r_1) and (q_2, r_2) in (2.8) and (2.9) are decoupled. Let (q, r) be the admissible pair satisfying

$$1/2 < 2/q = \delta(r) = n/(4p) < 1 \quad .$$

Let $0 < t_1 < t_2$. By the same estimates as in Sections 2 and 3 applied to the integral equation for u with initial time t_1, we estimate

$$\begin{aligned} y \quad &\equiv \quad \| u; L^q([t_1, t_2], H_r^1) \| \le C \| u(t_1); H^1 \| \\ &+ \quad C \| V \|_p \| u; L^q([t_1, t_2], H_r^1) \| \| u; L^k([t_1, t_2], L^r) \|^2 \end{aligned}$$

with $1/k + 1/q = 1/2$, so that $k > q$ since $q < 4$. We interpolate

$$\| u; L^k(L^r) \|^2 \le \| u; L^q(L^r) \|^{2-\lambda} \| u; L^\infty(L^r) \|^\lambda$$

with $0 < \lambda = 4 - 8p/n < 2$, so that

$$y \le M + C \| V \|_p \| u; L^\infty([t_1, t_2], L^r) \|^\lambda \ y^{3-\lambda} \quad . \tag{4.84}$$

By Lemma 4.6, $\| u; L^\infty([t_1, t_2], L^r) \|$ can be made arbitrarily small by taking t_1 sufficiently large, uniformly with respect to t_2. Furthermore for fixed t_1, y is a continuous (increasing) function of t_2, starting from zero for $t_2 = t_1$. It then follows from (4.84) that for t_1 sufficiently large y is bounded uniformly in t_2, namely that $u \in L^q([t_1, \infty), H_r^1)$. Plugging that result again into the integral equation yields that $u \in X^1(I\!R^+)$. The same argument holds for negative times.

Part (2). The fact that Ω_+ is a bijection of H^1 onto H^1 follows from the fact that any initial data $u(0) = u_0 \in H^1$ generates a (unique) solution $u \in X^1(I\!R)$ by Part (1) of this proposition, so that u has an asymptotic state $u_+ = \lim_{t \to \infty} \tilde{u}(t)$ by Proposition 3.2, part (1) and satisfies the equation (3.3) by Proposition 3.2 part (2) and therefore $u(0) = \Omega_+ \ u_+ \in \mathcal{R}(\Omega_+)$.

Boundedness of Ω_+ and Ω_+^{-1} follows from the conservation laws of the L^2 norm and of the energy, which together which (H1) (H3) imply that $\| u_0 \|_2 = \| u_+ \|_2$ and

$$\| \nabla u_0 \|_2 \leq \| \nabla u_+ \|_2 = \sqrt{2E} \leq \| u_0; H^1 \| \ + C \| u_0; H^1 \|^2 \quad .$$

Continuity of Ω_+ and Ω_+^{-1} follows from the corresponding properties in Propositions 2.1 and 3.1.

□

Remark 4.2. It is an unfortunate feature of the method that it does not provide an estimate of the norm of a solution u in $X^1(I\!R)$ in terms of the norm of $u(0)$ in H^1, or equivalently of the norm of $u_+ = \tilde{u}(\infty)$.

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